An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 40 (2000) 995–1002
www.elsevier.nl/locate/camwa

Regularized Auxiliary Problem Principle for Variational Inequalities

J. BAASANSUREN AND A. A. KHAN*
Fachbereich Mathematik, Universität Kaiserslautern
Postfach 3049, D-67653, Kaiserslautern, Germany
<basca><khan>@mathematik.uni-kl.de

Dedicated to the blessed memory of Professor Werner Oettli

(Received December 1999; accepted January 2000)

Abstract—This paper is devoted to the unification of the auxiliary problem principle and the principle of iterative regularization for the approximate solution of variational inequalities. This unification permits the extension of the principle of iterative regularization to real reflexive Banach spaces. On the basis of this unification, an algorithm is proposed and the strong convergence of the proposed algorithm is depicted. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords—Variational inequality, Iterative regularization, Auxiliary problem principle.

1. INTRODUCTION

Throughout the paper, unless the contrary is stated, $\mathcal{X}$ denotes a real reflexive Banach space and $\mathcal{X}^*$ represents the topological dual of $\mathcal{X}$; $\langle\cdot,\cdot\rangle$ the associated pairing and $\|\cdot\|$ stands for the norm in $\mathcal{X}$. Let $\Omega \subseteq \mathcal{X}$ be nonempty, closed, and convex. Consider the (nonlinear) operator $\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{X}^*$ and the functional $\varphi : \mathcal{X} \longrightarrow \Re$. Let $f \in \mathcal{X}^*$ be arbitrary.

The present study is concerned with the following problem: find $x \in \Omega$ such that

$$\langle \mathcal{F}x - f,\ z - x\rangle \geq \varphi(x) - \varphi(z), \qquad \forall\, z \in \Omega. \tag{1}$$

The above problem is referred to as a variational inequality (for short, VI) and any element $x \in \Omega$ satisfying the above conditions is said to be a solution to (1). We shall denote by $\mathcal{S}(\mathcal{F}, f, \varphi)$ the set of all solutions to (1).

In recent years, the theory of variational inequalities has emerged as an important branch of pure, applied, and industrial mathematics. This theory provides us with a convenient mathematical apparatus for uniformly studying a wide range of problems arising in diverse fields such as structural mechanics, fluid flows through porous media, elasticity, economics, local and global optimization, finance, etc. (cf. [1–12] and the references therein).

We are grateful to Professor Dr. R. Tichatschke (University of Trier, Germany) for his insightful observations, remarks, and corrections made to improve the paper.

*Current address: Institute of Applied Mathematics, University of Erlangen-Nürnberg, Martensstr. 3, 91058 Erlangen, Germany.

PII: S0898-1221(00)00214-5

In the process of development of the theory of variational inequalities, two topics, namely, the regularization methods and the numerical methods have been always of great importance. The importance of the numerical methods is well-understood. However, the regularization methods deserve a word of explanation. These are the methods designed for the study of ill-posed problems, and it is well known that variational inequalities belong to this class. So far regularization methods have been quite successful in dealing with the absence of the conditions, such as coerciveness and/or strong monotonicity. These conditions are essential for the existence of solution(s) of variational inequalities, but not satisfied by the data for some of the most important practical problems. Although there have been successful attempts to couple the ideas behind the regularization methods with the numerical methods, the treatment is rather limited to a shorter class of variational inequalities. For example, one of the most efficient of such methods, the so-called principle of iterative regularization (for short, PIR) has been designed for variational inequalities of the first kind (cf. $\varphi = 0$, in (1)) and needs the Hilbertian structure for the problem formulation (cf. [1]).

As far as the numerical methods are concerned, one of the most efficient method is the so-called auxiliary problem principle (for short, APP) which was introduced by Cohen [2] to solve the problems of convex programming and further extended for variational inequalities by Cohen [3]. The idea of the APP is to solve a minimization problem, which is built around an auxiliary function, and is equivalent to the variational inequality under consideration. This approach has several advantages over the co-existing projection-type methods. Namely, this approach is applicable for a very general form of the variational inequalities, it is free from the requirement of the Hilbertian structure, and there is a large choice available for the auxiliary functions.

In recent years, a great attention has been given to the APP, and as a consequence, many useful characteristics of this approach have been identified. For example, by considering a proximal auxiliary principle (for short, PAP) Kaplan-Tichatschke [8] has shown that the APP even generalize the so-called proximal methods. In [11], it has been shown that some modifications of the auxiliary function substantially improve the range for applications of the APP. In a series of papers, Cohen and his co-workers (see [5] and cited references therein) have considered the coupling of the APP with Yosida regularization.

In this present work, we study the unification of the APP and the PIR. This unification permits us to develop an iterative algorithm which converges strongly to the solution of (1) in the setting of a real reflexive Banach space. We do not require for any extension of the monotonicity assumption such as uniform or strong monotonicity.

The rest of the paper is organised as follows. In the next section, we recall some results to be used throughout the paper and discuss very briefly the concept of the regularization. Section 3 presents the unification of the APP and the PIR. On the basis of this unification, we propose a strongly convergent algorithm. An example is also given for the possible choices of the parameters. The paper concludes with some remarks concerning the approach.

2. PRELIMINARIES

In order to make this paper self contained, we briefly set forth below some basic definitions and results which we use here. For more details, the reader is referred to [1].

Let $\mathcal{Z}$ denote a real reflexive Banach space, $\mathcal{Z}^*$ the topological dual of $\mathcal{Z}$, $\langle\cdot,\cdot\rangle_{\mathcal{Z}}$ the associated pairing, and $\|\cdot\|_{\mathcal{Z}}$ norm in $\mathcal{B}$.

DEFINITION 2.1. *Let $\mathcal{T} : \mathcal{Z} \longrightarrow \mathcal{Z}^*$ and $x, z \in \mathcal{Z}$ be arbitrary. The operator $\mathcal{T}$ is said to be*

(i) *monotone, iff*

$$\langle \mathcal{T}x - \mathcal{T}z, x - z\rangle \geq 0;$$

(ii) *strongly monotone, iff there exists a constant $m > 0$ such that*

$$\langle \mathcal{T}x - \mathcal{T}z, x - z\rangle \geq m\|x - z\|^2;$$

(iii) *Lipschitz continuous, iff there exists a constant* $L > 0$ *such that*

$$\|\mathcal{T}x - \mathcal{T}z\| \le L\|x - z\|.$$

The following result can be found in [2].

THEOREM 2.1. *Let* $\mathcal{K} \subseteq \mathcal{Z}$ *be nonempty, closed, and convex,* $\mathcal{T} : \mathcal{Z} \longrightarrow \mathcal{Z}^*$ *be single-valued, monotone, and hemicontinuous,* ϕ *be a proper, convex, and lower-semicontinuous (for short, l.s.c.) functional and* $g \in \mathcal{Z}^*$ *be arbitrary. Assume that either the set* $\mathcal{K}$ *is bounded or the following coerciveness condition holds. There exists* $x_0 \in \mathcal{K}$ *such that* $\phi(x_0) < \infty$ *and*

$$\lim_{\|x\| \to \infty} \frac{\langle \mathcal{T}x, x - x_0\rangle + \phi(x)}{\|x\|} = +\infty, \qquad \forall\, x \in \mathcal{K}.$$

Then, the set of solutions to the problem: find $x \in \mathcal{K}$ *such that*

$$\langle \mathcal{T}x - g, z - x\rangle \ge \phi(x) - \phi(z), \qquad \forall\, z \in \mathcal{K},$$

is nonempty. Moreover, the solution set will be singleton, if the coerciveness condition is replaced by a stronger hypothesis, namely, the operator $\mathcal{T}$ *is strongly monotone.*

REMARK 2.1. Under the hypothesis of Theorem 2.1, the solution set to the above problem is closed and convex.

Consider the following regularized variational inequality (for short, RVI). For $\epsilon > 0$ find $x_\epsilon \in \Omega$ such that

$$\langle \mathcal{F}x_\epsilon + \epsilon\mathcal{R}x_\epsilon - f, z - x_\epsilon\rangle \ge \varphi(x_\epsilon) - \varphi(z), \qquad \forall\, z \in \Omega, \tag{2}$$

where $\mathcal{R} : \mathcal{X} \longrightarrow \mathcal{X}^*$ is (single-valued) hemicontinuous and strongly monotone.

In the above RVI, the operator $\mathcal{R}$ is termed as the regularizing operator and ϵ is known as the regularizing parameter. For more details of the topic, the reader is referred to [1].

The following result highlights the relationship between (1) and (2).

THEOREM 2.2. *Let* $\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{X}^*$ *be single-valued, monotone, and hemicontinuous,* $\mathcal{R} : \mathcal{X} \longrightarrow \mathcal{X}^*$ *be single-valued, hemicontinuous, and strongly monotone, the functional* φ *be proper, convex, and l.s.c., and* $\operatorname{int}\mathcal{D}(\varphi) \ne \emptyset$. *Then*

(i) *for each* $\epsilon > 0$, *(2) possesses a unique solution* x_ϵ.
(ii) *If the set* $\mathcal{S}(\mathcal{F}, f, \varphi) \ne \emptyset$, *then the following estimate holds:*

$$\lim_{\epsilon \to 0} \|x_\epsilon - x^*\| = 0, \tag{3}$$

where $x^* \in \mathcal{S}(\mathcal{F}, f, \varphi)$ *is the unique solution to the problem: find* $x \in \mathcal{S}(\mathcal{F}, f, \varphi)$ *such that*

$$\langle \mathcal{R}x, z - x\rangle \ge 0, \qquad \forall\, z \in \mathcal{S}(\mathcal{F}, f, \varphi). \tag{4}$$

(iii) *If* x_{ϵ_m} *and* x_{ϵ_n} *are the two solutions to (2) corresponding to the regularizing parameters* ϵ_m *and* ϵ_n, *respectively, then the following estimate is valid:*

$$\|x_{\epsilon_m} - x_{\epsilon_n}\| \le K\, \frac{|\epsilon_m - \epsilon_n|}{\epsilon_m}, \tag{5}$$

where K *is certain constant.*

3. MAIN RESULTS

Let $\mathcal{A} : \mathcal{X} \longrightarrow \Re$ be proper, convex, and Gâteaux differentiable, and $\mathcal{A}'$ be its Gâteaux derivative.

For an approximate solution of (1), we proceed as follows. We begin with an initial guess x^0 and initial parameters ϵ_0 and α_0, and solve the following problem: find $x \in \Omega$ such that

$$\min_{x\in\Omega} \mathcal{A}x + \left\langle \alpha_0 \left(\mathcal{F}x^0 + \epsilon_0 \mathcal{R}x^0\right) - \mathcal{A}'x^0, x\right\rangle + \alpha_0 \varphi\left(x^0\right). \tag{6}$$

Let the functional $\mathcal{A}$ be so chosen that the above minimization problem is uniquely solvable. We denote the (unique) solution by x^1 and continue further by replacing ϵ_0, α_0, and x^0 by ϵ_1, α_1, and x^1, respectively. Of course, in order to be in the framework of Theorem 2.2, we need to choose $\epsilon_1 < \epsilon_0$. Since (6) is emerging from (3), it is clear from the above-mentioned iteration process that as the number of iterations grows the proximity between (1) and (3) increases.

More precisely, we consider the following algorithm.

ALGORITHM.

(i) At $k = 0$ start with x^0, ϵ_0, and α_0.

(ii) At step $k = n$ solve the following problem: find $x \in \Omega$ such that

$$\min_{x\in\Omega} \mathcal{A}x + \langle \alpha_n \left(\mathcal{F}x^n + \epsilon_n \mathcal{R}x^n\right) - \mathcal{A}'x^n, x\rangle + \alpha_n \varphi\left(x^n\right). \tag{7}$$

Let x^{n+1} be the solution.

(iii) Stop if $\|x^{n+1} - x^n\|$ is below some threshold. Otherwise, go back to the previous step. Some particular cases of the above algorithm can be found in [2–5,8,10–12].

For the sequences $\{\epsilon_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$, we make the following assumption.

ASSUMPTION A. $0 < \alpha_n \le 1$; $0 < \epsilon_{n+1} \le \epsilon_n \le 1$; $\epsilon_n \to 0$ as $n \to \infty$,

$$\sum_{n=0}^{\infty} \alpha_n \epsilon_n = \infty; \qquad \sum_{n=0}^{\infty} \alpha_n^2 < \infty,$$

$$\text{for } \lambda_n = \frac{\epsilon_n - \epsilon_{n+1}}{\epsilon_n}, \qquad \sum_{n=0}^{\infty} \lambda_n^2 (\alpha_n \epsilon_n)^{-1} < \infty.$$

THEOREM 3.1. *Besides the hypotheses of Theorem 2.2, assume that the functional $\mathcal{A}$ is proper, convex, and Gâteaux differentiable and its Gâteaux derivative $\mathcal{A}'$ is strongly monotone and Lipschitz continuous. Then, for every $n \in \mathcal{N}$, there exists a unique solution x^{n+1} to (7). Moreover, if Assumption A holds, $\mathcal{F}$ and $\mathcal{R}$ are Lipschitz continuous and $\mathcal{S}(\mathcal{F}, f, \varphi) \neq \emptyset$, then*

$$\lim_{n\to\infty} \left\|x^{n+1} - x^*\right\| = 0,$$

where the element x^* is the unique solution to (4).

PROOF. Without any loss to the generality, we assume that $f = 0$. The existence of x^{n+1} follows from Theorem 2.2. In view of the obvious triangle inequality

$$\left\|x^{n+1} - x^*\right\| \le \left\|x^{n+1} - x_{\epsilon_n}\right\| + \left\|x_{\epsilon_n} - x^*\right\|$$

and (3), it is sufficient to show that

$$\lim_{n\to\infty} \left\|x^{n+1} - x_{\epsilon_n}\right\| = 0. \tag{8}$$

For this, we introduce the functional

$$\Phi(*) = \mathcal{A}(\bar{*}) - \mathcal{A}(*) - \langle \mathcal{A}'(*), \bar{*} - * \rangle, \tag{9}$$

where $(*)$ is the approximate solution of an RVI via (7) with $(\bar{*})$ as its unique regularized solution.

In view of the hypotheses on the functional $\mathcal{A} : \mathcal{X} \longrightarrow \Re$ and on its derivative $\mathcal{A}'$, the following estimates hold (cf. [12]):

$$\mathcal{A}x - \mathcal{A}z \geq \langle \mathcal{A}'z, x - z \rangle + \left(\frac{m}{2}\right) \|x - z\|^2 \tag{10}$$

and

$$\mathcal{A}x - \mathcal{A}z \geq \langle \mathcal{A}'x, x - z \rangle - \left(\frac{M}{2}\right) \|x - z\|^2, \tag{11}$$

where m and M are the modulus of strong monotonicity and Lipschitz continuity for $\mathcal{A}'$, respectively.

In light of (10) and (11), the functional $\Phi(\cdot)$ satisfies

$$\left(\frac{m}{2}\right) \|* - \bar{*}\|^2 \leq \Phi(*) \leq \left(\frac{M}{2}\right) \|* - \bar{*}\|^2. \tag{12}$$

We have the following specifications for the functional $\Phi(\cdot)$.

Iteration Level. $k = n - 1$.

$$\Phi(x^n) = \mathcal{A}x_{\epsilon_{n-1}} - \mathcal{A}x^n - \left\langle \mathcal{A}'x^n, x_{\epsilon_{n-1}} - x^n \right\rangle. \tag{13}$$

Iteration Level. $k = n$.

$$\Phi\left(x^{n+1}\right) = \mathcal{A}x_{\epsilon_n} - \mathcal{A}x^{n+1} - \left\langle \mathcal{A}'x^{n+1}, x_{\epsilon_n} - x^{n+1} \right\rangle. \tag{14}$$

We make the following notation:

$$\Delta^n = \left\| x^n - x_{\epsilon_{n-1}} \right\|.$$

In order to prove (8), we first show that the sequence $\{\Delta^n\}_{n=1}^{\infty}$ is uniformly bounded.

For this, let us analyze the difference

$$\begin{aligned}
\Phi(x^n) - \Phi\left(x^{n+1}\right) &= \left\{\mathcal{A}x_{\epsilon_{n-1}} - \mathcal{A}x^n - \left\langle \mathcal{A}'x^n, x_{\epsilon_{n-1}} - x^n \right\rangle\right\} \\
&\quad - \left\{\mathcal{A}x_{\epsilon_n} - \mathcal{A}x^{n+1} - \left\langle \mathcal{A}'x^{n+1}, x_{\epsilon_n} - x^{n+1} \right\rangle\right\} \\
&= \mathcal{A}x_{\epsilon_{n-1}} - \mathcal{A}x_{\epsilon_n} + \left\langle \mathcal{A}'x^{n+1}, x_{\epsilon_n} - x^{n+1} \right\rangle \\
&\quad + \mathcal{A}x^{n+1} - \mathcal{A}x^n - \left\langle \mathcal{A}'x^n, x_{\epsilon_{n-1}} - x^n \right\rangle \\
&= \mathcal{A}x_{\epsilon_{n-1}} - \mathcal{A}x_{\epsilon_n} + \left\langle \mathcal{A}'x^{n+1}, x_{\epsilon_n} - x^{n+1} \right\rangle \\
&\quad + \mathcal{A}x^{n+1} - \mathcal{A}x^n - \left\langle \mathcal{A}'x^n, x^{n+1} - x^n \right\rangle \\
&\quad - \left\langle \mathcal{A}'x^n, x_{\epsilon_{n-1}} - x^{n+1} \right\rangle.
\end{aligned}$$

Now making use of (10) and (11) to the above equation, we obtain

$$\begin{aligned}
\Phi(x^n) - \Phi\left(x^{n+1}\right) &\geq \left(\frac{m}{2}\right) \left\|x^n - x^{n+1}\right\|^2 - \left(\frac{M}{2}\right) \left\|x_{\epsilon_{n-1}} - x_{\epsilon_n}\right\|^2 \\
+ \left\langle \mathcal{A}'x_{\epsilon_{n-1}} - \mathcal{A}'x^n, x_{\epsilon_{n-1}} - x_{\epsilon_n} \right\rangle &+ \left\langle \mathcal{A}'x^{n+1} - \mathcal{A}'x^n, x_{\epsilon_n} - x^{n+1} \right\rangle.
\end{aligned} \tag{15}$$

We intend to find a suitable replacement for the last term of the above inequality. For this, we turn to (2). Setting $z = x^{n+1}$ in (2) and $\epsilon = \epsilon_n$, we obtain

$$\varphi\left(x^{n+1}\right) - \varphi(x_{\epsilon_n}) \geq \left\langle \mathcal{F}x_{\epsilon_n} + \epsilon_n \mathcal{R}x_{\epsilon_n},\, x_{\epsilon_n} - x^{n+1} \right\rangle.$$

Further, setting $z = x_{\epsilon_n}$ in (7), we obtain

$$\left\langle \mathcal{A}'x^{n+1} + \alpha_n\left(\mathcal{F}x^n + \epsilon_n \mathcal{R}x^n\right) - \mathcal{A}'x^n,\, x_{\epsilon_n} - x^{n+1} \right\rangle \geq \alpha_n\left(\varphi\left(x^{n+1}\right) - \varphi\left(x_{\epsilon_n}\right)\right).$$

We sum up the above two inequalities to obtain

$$\left\langle \mathcal{A}'x^{n+1} - \mathcal{A}'x^n, x_{\epsilon_n} - x^{n+1} \right\rangle \geq \alpha_n \left\langle \left(\mathcal{F}x_{\epsilon_n} + \epsilon_n \mathcal{R}x_{\epsilon_n}\right) - \left(\mathcal{F}x^n + \epsilon_n \mathcal{R}x^n\right), x_{\epsilon_n} - x^{n+1} \right\rangle. \quad (16)$$

Let $L = (L_1 + \epsilon_0 L_2)$, where L_1 and L_2 are the modulus of the Lipschitz continuity for the operators $\mathcal{F}$ and $\mathcal{R}$, respectively. Then for all $x_1, x_2 \in \Omega$, we have the following estimate:

$$\left\|\left(\mathcal{F}x_1 + \epsilon_n R x_1\right) - \left(\mathcal{F}x_2 + \epsilon_n R x_2\right)\right\| \leq L\left\|x_1 - x_2\right\|, \qquad \forall\, x_1, x_2 \in \Omega. \quad (17)$$

Now, combining (15) and (16), we obtain

$$\Phi\left(x^n\right) - \Phi\left(x^{n+1}\right) \geq T_1 + T_2 + T_3 + T_4, \quad (18)$$

where

$$\begin{aligned}
T_1 &= \alpha_n \left\langle \left(\mathcal{F}x^n + \epsilon_n \mathcal{R}x^n\right) - \left(\mathcal{F}x_{\epsilon_n} + \epsilon_n \mathcal{R}x_{\epsilon_n}\right), x^{n+1} - x^n \right\rangle + \left(\frac{m}{2}\right)\left\|x^n - x^{n+1}\right\|^2 \\
&= \alpha_n \left\langle \left(\mathcal{F}x^n + \epsilon_n \mathcal{R}x^n\right) - \left(\mathcal{F}x_{\epsilon_{n-1}} + \epsilon_n \mathcal{R}x_{\epsilon_{n-1}}\right), x^{n+1} - x^n \right\rangle + \left(\frac{m}{2}\right)\left\|x^n - x^{n+1}\right\|^2 \\
&\quad + \alpha_n \left\langle \left(\mathcal{F}x_{\epsilon_{n-1}} + \epsilon_n \mathcal{R}x_{\epsilon_{n-1}}\right) - \left(\mathcal{F}x_{\epsilon_n} + \epsilon_n \mathcal{R}x_{\epsilon_n}\right), x^{n+1} - x^n \right\rangle \\
&\geq \left(\frac{m}{2}\right)\left\|x^n - x^{n+1}\right\|^2 - \frac{\alpha_n^2 L^2}{m}\left\|x^n - x_{\epsilon_{n-1}}\right\|^2 - \left(\frac{m}{4}\right)\left\|x^n - x^{n+1}\right\|^2 \\
&\quad - \frac{\alpha_n^2 L^2}{m}\left\|x_{\epsilon_n} - x_{\epsilon_{n-1}}\right\|^2 - \left(\frac{m}{4}\right)\left\|x^n - x^{n+1}\right\|^2 \\
&\geq -\frac{\alpha_n^2 L^2}{m}\left\|x^n - x_{\epsilon_{n-1}}\right\|^2 - \frac{L^2}{m}\left\|x_{\epsilon_n} - x_{\epsilon_{n-1}}\right\|^2; \\
T_2 &= \alpha_n \left\langle \left(\mathcal{F}x^n + \epsilon_n \mathcal{R}x^n\right) - \left(\mathcal{F}x_{\epsilon_n} + \epsilon_n \mathcal{R}x_{\epsilon_n}\right), x^n - x_{\epsilon_{n-1}} \right\rangle \\
&= \alpha_n \left\langle \left(\mathcal{F}x^n + \epsilon_n \mathcal{R}x^n\right) - \left(\mathcal{F}x_{\epsilon_{n-1}} + \epsilon_n \mathcal{R}x_{\epsilon_{n-1}}\right), x^n - x_{\epsilon_{n-1}} \right\rangle \\
&\quad + \alpha_n \left\langle \left(\mathcal{F}x_{\epsilon_{n-1}} + \epsilon_n \mathcal{R}\left(x_{\epsilon_{n-1}}\right)\right) - \left(\mathcal{F}x_{\epsilon_n} + \epsilon_n \mathcal{R}x_{\epsilon_n}\right), x^n - x_{\epsilon_{n-1}} \right\rangle \\
&\geq r\alpha_n\epsilon_n \left\|x^n - x_{\epsilon_{n-1}}\right\|^2 - \frac{\alpha_n^2 L^2}{M}\left\|x^n - x_{\epsilon_{n-1}}\right\| - \frac{M}{4}\left\|x_{\epsilon_n} - x_{\epsilon_{n-1}}\right\|^2; \\
T_3 &= \alpha_n \left\langle \left(\mathcal{F}x^n + \epsilon_n \mathcal{R}x^n\right) - \left(\mathcal{F}x_{\epsilon_{n-1}} + \epsilon_n \mathcal{R}x_{\epsilon_{n-1}}\right), x_{\epsilon_{n-1}} - x_{\epsilon_n} \right\rangle - \left(\frac{M}{2}\right)\left\|x_{\epsilon_n} - x_{\epsilon_{n-1}}\right\|^2 \\
&= \alpha_n \left\langle \left(\mathcal{F}x^n + \epsilon_n \mathcal{R}x^n\right) - \left(\mathcal{F}x_{\epsilon_{n-1}} + \epsilon_n \mathcal{R}x_{\epsilon_{n-1}}\right), x_{\epsilon_{n-1}} - x_{\epsilon_n} \right\rangle \\
&\quad + \alpha_n \left\langle \left(F_{\epsilon_{n-1}} + \epsilon_n \mathcal{R}x_{\epsilon_{n-1}}\right) - \left(\mathcal{F}x_{\epsilon_n} + \epsilon_n \mathcal{R}x_{\epsilon_n}\right), x_{\epsilon_{n-1}} - x_{\epsilon_n} \right\rangle - \left(\frac{M}{2}\right)\left\|x_{\epsilon_n} - x_{\epsilon_{n-1}}\right\|^2 \\
&\geq r\alpha_n\epsilon_n \left\|x_{\epsilon_{n-1}} - x_{\epsilon_n}\right\|^2 - \frac{3M}{4}\left\|x_{\epsilon_{n-1}} - x_{\epsilon_n}\right\|^2 - \frac{L^2\alpha_n^2}{M}\left\|x^n - x_{\epsilon_{n-1}}\right\|^2; \\
T_4 &= \left\langle \mathcal{A}'x_{\epsilon_{n-1}} - \mathcal{A}'x^n, x_{\epsilon_n} - x_{\epsilon_{n-1}} \right\rangle \\
&\geq -M\left\|x_{\epsilon_{n-1}} - x^n\right\|\left\|x_{\epsilon_{n-1}} - x_{\epsilon_n}\right\| \\
&\geq -c\alpha_n\epsilon_n \left\|x_{\epsilon_{n-1}} - x^n\right\|^2 - \frac{M^2\left\|x_{\epsilon_{n-1}} - x_{\epsilon_n}\right\|^2}{4c\alpha_n\epsilon_n}; \qquad \theta = r - c > 0.
\end{aligned}$$

Plugging the estimates for T_1, T_2, T_3, and T_4 in (18), we obtain

$$\begin{aligned}\Phi\left(x^n\right)-\Phi\left(x^{n+1}\right) &\geq \theta\alpha_n\epsilon_n\left\|x^n-x_{\epsilon_{n-1}}\right\|^2-\frac{(M+2m)L^2\alpha_n^2}{mM}\left\|x^n-x_{\epsilon_{n-1}}\right\|^2\\ &\quad-\frac{Mm+L^2}{m}\left\|x_{\epsilon_n}-x_{\epsilon_{n-1}}\right\|^2+r\alpha_n\epsilon_n\left\|x_{\epsilon_n}-x_{\epsilon_{n-1}}\right\|^2\\ &\geq \theta\alpha_n\epsilon_n\left\|x^n-x_{\epsilon_{n-1}}\right\|^2-\frac{(M+2m)L^2\alpha_n^2}{mM}\left\|x^n-x_{\epsilon_{n-1}}\right\|^2\\ &\quad-\frac{\left(mM+L^2\right)^2}{4rm^2}\frac{\left\|x_{\epsilon_n}-x_{\epsilon_{n-1}}\right\|^2}{\alpha_n\epsilon_n}.\end{aligned} \tag{19}$$

Therefore,

$$\Phi\left(x^{n+1}\right)\leq\Phi\left(x^n\right)+\left[-\theta\alpha_n\epsilon_n\left\|x^n-x_{\epsilon_{n-1}}\right\|^2+c_1\alpha_n^2\left\|x^n-x_{\epsilon_{n-1}}\right\|^2+c_2\frac{\left\|x_{\epsilon_n}-x_{\epsilon_{n-1}}\right\|^2}{\alpha_n\epsilon_n}\right],$$

where $c_1=(M+2m)L^2/mM$ and $c_2=(mM+L^2)^2/4rm^2$.

Considering analogue of the above inequality from $n=1$ to N, summing them side-by-side and using (5), we obtain

$$\left(\frac{m}{2}\right)\Delta_{n+1}^2\leq\left(\frac{M}{2}\right)\Delta_1^2+\sum_{n=1}^{N}\left[-\theta\alpha_n\epsilon_n\Delta_n^2+c_1\alpha_n^2\Delta_n^2+c_2\lambda_n^2\left(\alpha_n\epsilon_n\right)^{-1}\right]. \tag{20}$$

The above estimate, in view of Assumption A and Lemma 5 of [4] confirms the boundedness of the sequence $\{\Delta^n\}_{n=1}^{\infty}$.

Moreover, (20) in view of the boundedness of $\{\Delta^n\}_{n=1}^{\infty}$ gives the following estimate:

$$\sum_{n=1}^{\infty}\theta\alpha_n\epsilon_n\Delta_n^2<\infty.$$

This, in view of the divergence of the series $\sum_{n=1}^{\infty}\alpha_n\epsilon_n$ and the above observations, confirms that

$$\{\Delta^n\}_{n=1}^{\infty}\longrightarrow 0.$$

This completes the proof. ∎

EXAMPLE. Let $1/2<\kappa_1<1$, $\kappa_2>0$, $\kappa_1+\kappa_2<1$.

Then for the sequences of the form

$$\alpha_n=n^{-\kappa_1}\qquad\text{and}\qquad\epsilon_n=n^{-\kappa_2},$$

Assumption A is fulfilled.

4. CONCLUDING REMARKS

In this paper, we have studied the unification of the APP and the PIR. Such a unification is advantageous due to the possibility of cross-fertilization of the ideas behind these concepts. The remarkable formulation of the auxiliary problem permits us to introduce an iterative scheme in the setting of Banach spaces. The PIR allows us to establish the strong convergence without requiring any sort of extension to the monotonicity of the involved operator, like uniform or strong monotonicity. We only demand that the problem under consideration is solvable.

The present approach can be extended in many direction. For example, it is possible to consider more general class of variational inequalities. At this point, we would like to mention a recent survey article [10] where one can find more general formulations. Here we have confined our attention to the auxiliary function of Cohen [2]. However, in view of the recent results, it is possible to consider more general auxiliary functions. Though the minimization formulation is an advantage, it might also be of interest to develop a general scheme with nonsmooth auxiliary functions.

From the application point of view, it would be of interest to consider convex sets of the form

$$\Omega = [x \in \Omega_0 : g_i(x) \leq 0,\ i = 1, 2, \dots, m;\ g_i(x) = 0,\ i = m+1, 2, \dots, s],$$

where Ω_0 is any given convex set and the functions g_i specify certain constraints. In this situation, a sort of iterative-regularization-penalization is needed.

REFERENCES

1. A.B. Bakushinskii and A.V. Goncharskii, *Iterative Methods for Ill-Posed Problems*, Kluwer Academic, Dordrecht, (1994).
2. G. Cohen, Auxiliary problem principle and decomposition of optimization problems, *J. Optim. Theory Appl.* **32**, 277–305, (1980).
3. G. Cohen, Auxiliary problem principle extended to variational inequalities, *J. Optim. Theory Appl.* **59**, 325–333, (1980).
4. G. Cohen and D.L. Zhu, Decomposition coordination methods in large scale optimization problems: The nondifferentiable case and the use of augmented lagrangians, In *Advances in Large Scale Systems*, (Edited by J.B. Cruz, Jr.), pp. 203–266, Jai Press, Greenwich, CT, (1984).
5. N.E. Farouq and G. Cohen, Progressive regularization of variational inequalities and decomposition algorithms, *J. Optim. Theory Appl.* **97**, 407–433, (1998).
6. F. Giannessi, On connectedness among separation, penalization and regularization for variational inequalities with point-to-set operators, *Suppl. Rendi. Circolo Matemat. Paler. Serie II-Num.* **48**, 137–145, (1997).
7. G. Isac, Tikhonov regularization and the complementarity problem in Hilbert spaces, *J. Math. Anal. and Appl.* **174**, 53–66, (1993).
8. A. Kaplan and R. Tichatschke, Auxiliary problem principle and proximal point methods, Preprint, (June 1999).
9. D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, (1980).
10. M.A. Noor, K.I. Noor and T.M. Rassias, Some aspects of variational inequalities, *J. Comp. Appl. Math.* **47**, 285–312, (1993).
11. G. Salmon, V.H. Nguyen and J.J. Strodiat, Coupling of the auxiliary problem principle and epiconvergence theory for solving general variational inequalities, *J. Optim. Theory Appl.* **104**, (2000) (to appear).
12. D.L. Zhu and P. Marcotte, Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities, *SIAM J. Optim.* **6**, 714–726, (1996).